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ON A VARIATIONAL FORMULA AND ITS APPLICATION TO CONTACT PROBLEMS OF ELASTICITY THEORY*

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A variational formula is obtained for the spatial contact problem (CP) of the theory of elasticity. This formula determines the variation of the normal stress on the contact area caused by a variation in the contact area outline. The efficiency of the variational formula is shown for constructing an asymptotic expansion for the spatial CP with a continuous line of separation of the boundary conditions. The solution of the CP is considered in detail for an elastic half-space when the contact area differs slightly from a circle. A survey of applications of asymptotic methods to the CP of elasticity theory is given in /1, 2/. Solutions of the spatial CP with a continuous line of separation of the boundary conditions obtained by other methods are presented in /3-6/.

1. We use a rectangular coordinates system x_1, x_2, x_3. We consider a linearly elastic body occupying the simply connected volume V. Let the surface O bounding this volume consist of a certain surface O_1 and a plane surface O_0 whose equation is $x_3 = 0$. A stiff cylindrical stamp of arbitrary section is impressed in the plane surface of the body O_0. The stamp base has the shape of a convex surface. The equation of the base surface of the stamp has the form

$$x_3 = -f_0(x_1, x_2) \tag{1.1}$$

The plane surface O_0 here will be separated into two parts: the contact area (CA) O_3 and the surface O_2.

We consider the magnitude and line of action of the force P such that the contact area O_3 coincides with the transverse section of a stiff cylinder (stamp). The surface O_2 is stress-free while a static boundary condition is given on the surface O_1. The shear stress on the CA O_3 equals zero. The projection of the displacement vector on the x_3 axis within the limits of the CA will be expressed by the formula

$$u_3(x_1, x_2, 0) = c - \beta_2 x_1 + \beta_1 x_2 - f_0(x_1, x_2), \quad (x_1, x_2) \subset O_3$$

We denote the boundary outline of the CA by Γ_0. We magnify the size of the CA by displacing the outline Γ_0 to the nearby position Γ. We here direct the variation $\delta n(M)$ at each point M of the CA outline along the outer normal to the curve Γ_0. In this case the following relationship /7/ will hold

$$\int_{\Gamma_0} K_1^2(M)\,\delta n(M)\,ds = -\frac{2\mu}{\pi(1-\nu)} \iint_{O_3} u_3(Q)\,\delta_n\sigma_{33}(Q)\,dS \tag{1.2}$$
$$M \in \Gamma_0, \quad Q \in O_3$$

where K_1 is the compressive normal stress intensity factor, μ is the shear modulus, ν is Poisson's ratio, $\sigma_{33}(Q)$ is the normal stress at the CA and $\delta_n\sigma_{33}$ is the variation of the stress σ_{33} caused by variation of the CA outline δn.

In a special case (for $K_1 = 0$) a formula follows from (1.2) which was applied successfully earlier /8, 9/ to solve the CP.

2. We consider two states of a given body by ascribing the respective superscripts (1) and (2). We also consider the total state for which

$$u_3 = u_3^{(1)}(Q) + u_3^{(2)}(Q), \quad \sigma_{33} = \sigma_{33}^{(1)}(Q) + \sigma_{33}^{(2)}(Q) \tag{2.1}$$
$$K_1 = K_1^{(1)}(M) + K_1^{(2)}(M), \quad M \in \Gamma_0, \quad Q \in O_3$$

Substituting (2.1) into (1.2) and using the theorem on reciprocity of work, we obtain

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$$\iint_{O_3} u_3^{(1)}\delta_n\sigma_{33}^{(2)}\,dS=\iint_{O_3} u_3^{(2)}\delta_n\sigma_{33}^{(1)}\,dS=-\frac{\pi(1-\nu)}{2\mu}\int_{\Gamma_0} K_1^{(1)}K_1^{(2)}\delta n\,ds \tag{2.2}$$

In the axial symmetry case this expression simplifies and takes the form

$$K_1^{(1)}K_1^{(2)}\delta a=-\frac{2\mu}{\pi(1-\nu)a}\int_0^a ru_3^{(1)}(r)\,\delta_n\sigma_{33}^{(2)}(r)\,dr \tag{2.3}$$

where a is the radius of a circle (the domain O_3), and δa is the variation of the CA radius. It hence follows that if $K_1^{(2)}$ and $\sigma_{33}^{(2)}(r)$ are known for some axisymmetric CA then the quantity $K_1^{(1)}$ can be found by means of (2.3) for any other axisymmetric CA.

Let us examine the special case of (2.2) when $u_3^{(1)}(Q)=U\delta(Q,Q_1)$, where $\delta(Q,Q_1)$ is the delta function. The intensity factor in this case is denoted by $K_1^{(1)}(M;Q_1)$. Then for $Q_1\in O_3$ we will have (we omit the superscripts (1) and (2)):

$$\delta_n\sigma_{33}(Q_1)=-\frac{\pi(1-\nu)}{2\mu}\int_{\Gamma_0} K_1^*(M;Q_1)K_1^{(0)}(M)\,\delta n(M)\,ds \tag{2.4}$$

where $K_1^*(M;Q_1)$ corresponds to $u_3(Q)=\delta(Q,Q_1)$ while $K_1^{(0)}(M)$ corresponds to a given displacement $u_3(Q)$ on the CA. Formula (2.4) expresses the variation of the normal stress on the CA caused by variation of the CA outline.

It follows from (1.2) that

$$\delta_n W=-\frac{\pi(1-\nu)}{2\mu}\int_{\Gamma_0} K_1^2(M)\,\delta n(M)\,ds \tag{2.5}$$

where W is the strain potential energy and $\delta_n W$ is the variation of the strain potential energy caused by variation of the CA outline.

We set $K_1=UK_1^*+K_1^{(0)}$ in (2.5). Then (2.4) can also be represented in the same form

$$\delta_n\sigma_{33}=\partial(\delta_n W)/\partial U|_{U=0} \tag{2.6}$$

In a certain sense relationship (2.6) is the analogue of a well-known formula in structural mechanics.

We will examine the special case of (2.4) when the CA O_3 is a circle of radius a while Γ_0 is a circle of the same radius. In this case by using cylindrical coordinates r,θ,z we obtain

$$\delta_n\sigma_z(r,\theta)=-\frac{\pi(1-\nu)a}{2\mu}\int_0^{2\pi} K_1^*(\varphi;r,\theta)K_1^{(0)}(\varphi)\,\delta n(\varphi)\,d\varphi \tag{2.7}$$

where φ is the polar angle corresponding to the point M.

The variational formulas (2.4) and (2.7) turned out to be quite effective for solving the spatial CP of elasticity theory with a complex line of separation of the boundary conditions.

3. We consider the CP for the elastic body from Sect.1. The CA boundary outline is denoted by Γ. Let the curve Γ deviate slightly from the circle of radius a (the curve Γ_0). The equation of the CA boundary outline in polar coordinates has the form

$$\rho=a[1+\varepsilon f(\varphi)],\quad \varepsilon\ll 1 \tag{3.1}$$

where $f(\varphi)$ is a certain piecewise-continuous function.

The normal stress distribution law $\sigma_z(r,\theta)$ on the CA S should be found by solving this problem. We will seek the unknown function $\sigma_z(r,\theta)$ in the form of an asymptotic expansion in a small parameter

$$\sigma_z(r,\theta)=\sigma_z^{(0)}(r,\theta)+\varepsilon\sigma_z^{(1)}(r,\theta)+O(\varepsilon^2) \tag{3.2}$$

where $\sigma_z^{(0)}(r,\theta)$ is the solution of the unperturbed problem (for a circular contact area). It is proposed to use the variational formula (2.7) to find the function $\sigma_z^{(1)}(r,\theta)$.

Taking into account that

$$\delta n(\varphi)=\varepsilon af(\varphi),\quad \delta_n\sigma_z(r,\theta)=\varepsilon\sigma_z^{(1)}(r,\theta)$$

and using (2.7), we find

$$\sigma_z^{(1)}(r,\theta) = -\frac{\pi(1-\nu)a^2}{2\mu}\int_0^{2\pi} K_1^*(\varphi; r,\theta) K_1^{(0)}(\varphi) f(\varphi)\, d\varphi \tag{3.3}$$

The quantities K_1^* and $K_1^{(0)}$ in (3.3) are determined for a circular CA of radius a.

Therefore, the asymptotic expansion (3.2) has been constructed. The CP for which the CA outline Γ is determined by (3.1), is here reduced to a problem with a circular CA.

For the solution obtained to be completed it is necessary to know the function $\sigma_z^{(0)}(r,\theta)$, $K_1^*(\varphi; r,\theta)$ and $K_1^{(0)}(\varphi)$. They can be determined if the form of the elastic body for which the CP is solved is made specific. The simplest results are obtained if the elastic body occupies a half-space. Consequently, we will consider the spatial CP below for an elastic half-space when the equation of the CA boundary outline has the form (3.1). To solve this problem it is necessary to have the solution of the corresponding problem with a circular CA. A number of methods has been proposed /1/, to solve the CP for an elastic half-space with a circular CA. However, the solution obtained below is most convenient for investigating the problem in question.

4. We consider the CP for an elastic isotropic half-space with a circular CA. We assume that friction forces on the CA not ot occur between the stamp and the half-space, an that there are no load on the half-space outside the stamp. The integral equation of this problem has the form

$$u_z(r,\theta) = -\frac{1-\nu}{2\pi\mu}\int_0^{2\pi} d\theta_1 \int_0^a \frac{\sigma_z(r_1,\theta_1)\, r_1\, dr_1}{[r^2 + r_1^2 - 2rr_1\cos(\theta-\theta_1)]^{1/2}} \tag{4.1}$$

We seek the solution of (4.1) in the form

$$\sigma_z(r,\theta) = \sum_{n=0}^{\infty} \sigma_{zn}(r)\cos n\theta \quad (r < a) \tag{4.2}$$

i.e., we consider the stress $\sigma_z(r,\theta)$ to be symmetric with respect to $\theta = 0$. In this case

$$u_z(r,\theta) = \sum_{n=0}^{\infty} u_{zn}(r)\cos n\theta \tag{4.3}$$

Substituting (4.2) and (4.3) into (4.1), we find after calculations

$$\sigma_{zn}(r) = \frac{2\mu r^{n-1}}{\pi(1-\nu)}\frac{d}{dr}\int_r^a \frac{t\psi_n(t)\, dt}{(t^2-r^2)^{1/2}} \tag{4.4}$$

$$\psi_n(t) = \frac{1}{t^{2n}}\frac{d}{dt}\int_0^t \frac{r_1^{n+1} u_{zn}(r_1)\, dr_1}{(t^2-r_1^2)^{1/2}}$$

Formulas (4.2)-(4.4) yield the solution of the CP for an elastic half-space in the presence of a circular CA. Furthermore, an expression can be obtained for the compressive stress intensity factor. We have

$$K_1(\varphi) = -\lim_{r\to a-0,\ \theta\to\varphi} [2^{1/2}(a-r)^{1/2}\sigma_z(r,\theta)]$$

Substituting (4.2) and (4.4) here, we find

$$K_1(\varphi) = \frac{2\mu}{\pi(1-\nu)}\sum_{n=0}^{\infty} a^{n-1/2}\psi_n(a)\cos n\varphi \tag{4.5}$$

Let us examine the special case when

$$u_z(r,\theta) = r^{-1}\delta(r-r_1)\delta(\theta)$$

where $\delta(r)$ is the delta function. In this case (4.5) yields

$$K_1^*(\varphi) = -\frac{\mu a^{1/2}}{\pi^2(1-\nu)(a^2-r_1^2)^{1/2}}\sum_{n=0}^{\infty}\varepsilon_n\left(\frac{r_1}{a}\right)^n\cos n\varphi$$

where $\varepsilon_0 = 1$, $\varepsilon_n = 2$ for $n > 0$. Summing the series, we finally obtain

$$K_1^*(\varphi) = -\frac{\mu a^{1/2}}{\pi^2(1-\nu)(a^2-r_1^2)^{1/2}(a^2+r_1^2-2ar_1\cos\varphi)} \tag{4.6}$$

5. It is now possible to return to the CP for an elastic half-space when the CA outline Γ is given by (3.1).

We will assume (for the simplified problem) that the CA S has two axes of symmetry and a force P is directed along the z axis that passes through the centre of gravity of the CA. In this case the stamp will be impressed strictly translationally (without rotation) into the elastic half-space and the displacement of the half-space surface within the limits of the CA will have the form

$$u_z(r, \theta) = c - f_0(r, \theta) \tag{5.1}$$

The solution of this problem is given by (3.2). The stress $\sigma_z^{(0)}(r, \theta)$ in (3.2) can be found by using expressions (4.2) and (4.4). To determine the function $\sigma_z^{(1)}(r, \theta)$, (3.3) should be used. By using (4.6) we finally find from (3.3)

$$\sigma_z^{(1)}(r, \theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(a^2-r^2)^{-1/2}\, v(a, \varphi)\, d\varphi}{a^2+r^2-2ar\cos(\varphi-\theta)} \tag{5.2}$$

$$v(a, \varphi) = a^{1/2} K_1^{(0)}(\varphi) f(\varphi)$$

The $K_1^{(0)}(\varphi)$ in this formula is defined for a circular CA depending on the function $u_z(r, \theta) = c - f_0(r, \theta)$. The formula (4.5) can be used to find the quantity $K_1^{(0)}$.

Therefore, the asymptotic expansion (3.2) that yields the solution of the CP under consideration is constructed.

We will examine certain properties of the function $\sigma_z^{(1)}(r, \theta)$. We multiply both sides of the first formula in (5.2) by $(a^2 - r^2)^{1/2}$. Then the integral on the right-hand side of the formula obtained is a Poisson integral. Therefore, the expression $(a^2 - r^2)^{1/2}\sigma_z^{(1)}(r, \theta)$ is a harmonic function within the circle $r < a$ and $v(a, \varphi)$ is the limit value of this function on the circumference of this circle, i.e.,

$$\lim_{r\to a,\ \theta\to\varphi} [(a^2 - r^2)^{1/2}\sigma_z^{(1)}(r, \theta)] = a^{1/2} K_1^{(0)}(\varphi) f(\varphi)$$

The function $v(a, \varphi)$ and therefore $f(\varphi)$ can also be an arbitrary piecewise-continuous function.

6. Let us consider examples of the use of the solution obtained. Let the stamp have a flat base. In this case $f_0(r, \theta) = 0$ and

$$\sigma_z^{(0)}(r, \theta) = -\frac{2\mu c}{\pi(1-\nu)(a^2-r^2)^{1/2}}, \quad K_1^{(0)}(\varphi) = \frac{2\mu c}{\pi(1-\nu)a^{1/2}} \tag{6.1}$$

Substituting (6.1) into (3.2) and (5.2), we find

$$\sigma_z(r, \theta) = -\frac{2\mu c}{\pi(1-\nu)(a^2-r^2)^{1/2}}\left[1 - \varepsilon\frac{a^2 F(r, \theta)}{a^2-r^2}\right] + O(\varepsilon^2) \tag{6.2}$$

$$F(r, \theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(a^2-r^2) f(\varphi)\, d\varphi}{a^2+r^2-2ar\cos(\varphi-\theta)}$$

It can be seen from an examination of the expression in the square brackets in the first formula in (6.2) that the assumption of smallness of the perturbations is violated near the critical point $r = a$. Consequently, the solution obtained is not uniformly suitable near the critical point. Uniform suitability of the expansion (6.2) can be restored by applying the method of deformed coordinates /10/.

We replace r in (6.2) by a slightly deformed coordinate r_0

$$r = r_0 + \Psi(r_0, \theta) \tag{6.3}$$

Substituting (6.3) into (6.2) we obtain

$$\sigma_z(r_0, \theta) = -\frac{2\mu c}{\pi(1-\nu)}(a^2 - r_0^2)^{-1/2}\left[1 + \varepsilon\frac{r_0\Psi(r_0, \theta)}{a^2-r_0^2} - \varepsilon\frac{a^2 F(r_0, \theta)}{a^2-r_0^2}\right] + O(\varepsilon^2) \tag{6.4}$$

By Lighthill's principle, higher-order approximations should not have a stronger singularity than the first approximation. On this basis we set

$$\Psi(r_0, \theta) = r_0 F(r_0, \theta) \tag{6.5}$$

Substituting (6.5) into (6.4) we will have

$$\sigma_z(r_0, \theta) = -\frac{2\mu c}{\pi(1-\nu)}[1-\varepsilon F(r_0,\theta)](a^2-r_0^2)^{-1/2}+O(\varepsilon^2)$$

Returning to the variable r we finally find

$$\sigma_z(r,\theta) = -\frac{2\mu c}{\pi(1-\nu)\{[1+2\varepsilon F(r,\theta)]a^2-r^2\}^{1/2}}+O(\varepsilon^2) \qquad (6.6)$$
$$r < a\,[1+\varepsilon f(\varphi)]$$

Formula (6.6) determines the stress σ_z on the CA S whose boundary outline is described by (3.1). The function $F(r,\theta)$ in (6.6) can be found by means of (6.2). Therefore expression (6.6) yields the complete solution of the CP in question for a stamp with a flat base.

A specific CA shape must be given, i.e., the function $f(\varphi)$, to show the course of the further calculations. For instance, let the CA boundary contour S be an ellipse with semi-axes $(1+\varepsilon)a$ and a. In this case,

$$f(\varphi) = \cos^2\varphi, \quad F(r,\theta) = (a^2 + r^2\cos 2\theta)/(2a^2) \qquad (6.7)$$

Changing to rectangular **coordinates** in (6.6) and using (6.7) we obtain

$$\sigma_{33}(x_1,x_2) = -\frac{2c\mu(1-\varepsilon/2)}{\pi(1-\nu)a}\left\{1-\frac{x_1^2}{[(1+\varepsilon)a]^2}-\frac{x_2^2}{a^2}\right\}^{-1/2}+O(\varepsilon^2) \qquad (6.8)$$
$$(x_1,x_2)\subset S$$

Furthermore, if the relationship

$$P = -\iint_S \sigma_{33}(x_1,x_2)\,dx_1\,dx_2$$

is taken into account, the depth of stamp impression can be determined

$$c = \frac{P(1-\nu)}{4\mu a}\left(1-\frac{\varepsilon}{2}\right)+O(\varepsilon^2) \qquad (6.9)$$

The exact solution of this problem, when the CA is an ellipse with semi-axes $(1+\varepsilon)a$ and a, has the form

$$\sigma_{33}(x_1,x_2) = -\frac{c\mu}{(1-\nu)a\mathbf{K}(k)}\left\{1-\frac{x_1^2}{[(1+\varepsilon)a]^2}-\frac{x_2^2}{a^2}\right\}^{-1/2} \qquad (6.10)$$
$$c = \frac{P(1-\nu)}{2\pi\mu(1+\varepsilon)a}\mathbf{K}(k), \quad k = \frac{[\varepsilon(2+\varepsilon)]^{1/2}}{1+\varepsilon}$$

where $\mathbf{K}(k)$ is the complete elliptic integral of the first kind.

If (6.10) is expanded in a power series in ε, we arrive at (6.8) and (6.9).

We will estimate the error that results from using (6.8). To do this we examine the quantity

$$\beta = \frac{2}{\pi}\left(1-\frac{\varepsilon}{2}\right)\mathbf{K}(k) \qquad (6.11)$$

For instance, $\beta = 0.985$ for $\varepsilon = 0.2$. Since the quantity β is the ratio of stresses σ_{33} calculated by means of (6.8) and (6.10), expression (6.8) gives an error of not more than 1.5% for $\varepsilon \leqslant 0.2$. Obviously, (6.6) indeed possesses the same accuracy for which (6.8) is a special case. But expression (6.6) is also applicable for other CA shapes for which there are no exact solutions.

Now let the CA boundary outline be determined by (3.1). We will consider the function $f(\varphi)$ in (3.1) to be an even function of φ. Expanding it in a Fourier series

$$f(\varphi) = \frac{A_0}{2}+\sum_{k=1}^{\infty}A_k\cos k\varphi, \quad A_k = \frac{1}{\pi}\int_{-\pi}^{\pi}f(\varphi)\cos k\varphi\,d\varphi, \quad k=0,1,2,\ldots$$

and substituting into the second formula of (6.2), we obtain after integrating

$$F(r,\theta) = \frac{1}{2}A_0+\sum_{k=1}^{\infty}A_k\left(\frac{r}{a}\right)^k\cos k\theta \qquad (6.12)$$

If (6.12) is substituted into (6.6), we obtain a formula to determine the stress σ_z on the CA whose boundary outline is given by (3.1).

Let us examine the special case of (3.1) when

$$\rho = a[1 + \varepsilon(1 + \cos 4\varphi)]$$

In this case $f(\varphi) = 1 + \cos 4\varphi$ and (6.12) yields

$$F(r, \theta) = 1 + (r/a)^4 \cos 4\theta$$

The examination of the case of a stamp with a non-planar base also does not present any difficulties in principle.

REFERENCES

1. GALIN L.A., Ed., Development of Contact Problem Theory in the USSR. Nauka, Moscow, 1976.
2. ALEKSANDROV V.M., An asymptotic method in contact problems of elasticity theory, PMM, 32, 4, 1968.
3. MOSSAKOVSKII V.I., KACHALOVSKAYA N.E. and GOLIKOVA S.S., Contact Problems of the Mathematical Theory of Elasticity. Naukova Dumka, Kiev, 1985.
4. MARTYNENKO M.D., Certain spatial problems of elasticity theory. Strength and Plasticity, Nauka, Moscow, 1971.
5. POPOV G.YA., Concentration of Elastic Stresses around Stamps, Slits, Fine Inclusions and Reinforcements. Nauka, Moscow, 1982.
6. GALIN L.A., Contact Problems of Elasticity and Viscoelasticity Theories. Nauka, Moscow, 1980.
7. BORODACHEV N.M., Application of the principle of minimum additional work for contact problems of elasticity theory, Prikl. Mekhan., 21, 9, 1985.
8. ALEKSANDROV V.M., Contact problems for half-spaces. Complex contact domain planforms. Development of the Theory of Contact Problems in the USSR, Nauka, Moscow, 1976.
9. PROSHIN V.S., On the question of boundedness of contact pressures on the outline of a stamp of elliptical planform interacting with an elastic layer, PMM, 48, 3, 1984.
10. VAN DYKE M., Perturbation Methods in Fluid Mechanics, Mir, Moscow, 1967.

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